

SEQUENCE-SINGULAR OPERATORS

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ABSTRACT. In this paper we study two types of collections of operators on a Banach space on the subject of forming operator ideals. One of the types allows us to construct an uncountable chain of closed ideals in each of the operator algebras $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q < \infty$, and $\mathcal{L}(\ell_1 \oplus c_0)$. This finishes answering a longstanding question of Pietsch.

1. INTRODUCTION

Fix a seminormalized basis $e = (e_n)$ for a Banach space E . Following Beanland and Freeman [BF11], we say that an operator $T \in \mathcal{L}(X, Y)$, X and Y Banach spaces, is **(e_n) -singular** just in case for every normalized basic sequence (x_n) in X , the image sequence (Tx_n) fails to dominate (e_n) . We denote by $\mathcal{WS}_{e, \omega_1}(X, Y)$ the class of all (e_n) -singular operators in $\mathcal{L}(X, Y)$. In [BF11, Proposition 2.8] the following interesting results were proved about class $\mathcal{WS}_{e, \omega_1}$ for certain nice choices of e .

- If $e = (e_n)$ denotes the canonical basis for c_0 then $\mathcal{WS}_{e, \omega_1} = \mathcal{K}$, the compact operators.
- If $e = (e_n)$ denotes the summing basis for c_0 then $\mathcal{WS}_{e, \omega_1} = \mathcal{W}$, the weakly compact operators.
- If $e = (e_n)$ denotes the canonical basis for ℓ_1 then $\mathcal{WS}_{e, \omega_1} = \mathcal{R}$, the Rosenthal operators.

(Recall that an operator $T \in \mathcal{L}(X, Y)$ is **Rosenthal** just in case for every bounded sequence (x_n) in X , (Tx_n) admits a weak Cauchy subsequence.) Each of these classes is a norm-closed operator ideal, and so it is natural to conjecture that class $\mathcal{WS}_{e, \omega_1}$ could also form an operator ideal for other nice choices of $e = (e_n)$. In particular, we might expect $\mathcal{WS}_{e, \omega_1}$ to be an operator ideal whenever $e = (e_n)$ is the canonical basis of ℓ_p , $1 < p < \infty$.

In the present paper, we show that the above conjecture is false, and indeed that for any $1 < p < \infty$ we can always choose spaces X and Y such that $\mathcal{WS}_{e, \omega_1}(X, Y)$ fails to be closed under addition when $e = (e_n)$ is the canonical basis for ℓ_p , $1 < p < \infty$.

Despite this, we might still be able to use them, or variants thereof, to investigate the closed ideal structure of the operator algebra $\mathcal{L}(X)$ for certain choices of X . Indeed, using a descriptive set-theoretic result from [BF11] together with a modification of the definition of $\mathcal{WS}_{e, \xi}$ to show that $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q < \infty$, and $\mathcal{L}(\ell_1 \oplus c_0)$ each admit an uncountable chain of closed ideals. This is especially significant since it represents the last ingredient needed to answer a longstanding open question of Pietsch ([Pi78, Problem 5.33]).

For the most part, all definitions and notation are standard, as are found, for instance, in [AK06]. However, we will restate some of the most important such here. Let \mathcal{J} be a subclass of the class \mathcal{L} of all continuous linear operators between Banach

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spaces, and if X and Y are Banach spaces then we write $\mathcal{J}(X, Y) = \mathcal{L}(X, Y) \cap \mathcal{J}$, a component. We say that \mathcal{J} has the **ideal property** whenever $BT A \in \mathcal{J}(W, Z)$ for all $A \in \mathcal{L}(W, X)$, $B \in \mathcal{L}(Y, Z)$, and $T \in \mathcal{J}(X, Y)$, and all Banach spaces W , X , Y , and Z . If in addition every component $\mathcal{J}(X, Y)$ is a linear subspace of $\mathcal{L}(X, Y)$ containing all the finite-rank operators therein, then \mathcal{J} is an **operator ideal**. We say that \mathcal{J} is norm-closed (closed under addition) whenever all its components $\mathcal{J}(X, Y)$ are norm-closed (closed under addition) in $\mathcal{L}(X, Y)$. Let us also borrow a piece of terminology from [Sc12]: If X and Y are Banach spaces, then a linear subspace \mathcal{J} of $\mathcal{L}(X, Y)$ is called a **subideal** just in case whenever $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$, and $T \in \mathcal{J}$, we have $BT A \in \mathcal{J}$. A subideal of $\mathcal{L}(X)$ is called, simply, an **ideal**. (For operator algebras, this coincides with the notion of an *ideal* in the algebraic sense.)

If M is an infinite subset of \mathbb{N} , then denote by $[M]$ the family of all infinite subsets of M , and denote by $[M]^{<\omega}$ the family of all finite subsets of M . For $n \in \mathbb{N}$ let $[M]^{\leq n} = \{A \in [M]^{<\omega} : \#A \leq n\}$, i.e. the family of all subsets of M of size $\leq n$. If \mathcal{F} is a subset of $[\mathbb{N}]^{<\omega}$ and $M = (m_i) \in [\mathbb{N}]$, then we define

$$\mathcal{F}(M) = \{(m_i)_{i \in E} : E \in \mathcal{F}\}.$$

If \mathcal{F} and \mathcal{G} are both subsets of $[\mathbb{N}]^{<\omega}$ then we define

$$\mathcal{F}[\mathcal{G}] = \left\{ \bigcup_{i=1}^n E_i : E_1 < \dots < E_n, E_i \in \mathcal{G} \forall i, (\min E_i)_{i=1}^n \in \mathcal{F} \right\}.$$

Let us now define the **Schreier families**. These are denoted \mathcal{S}_ξ for each countable ordinal $0 \leq \xi < \omega_1$, and we must define them as follows. For $\xi = 0$, we put $\mathcal{S}_0 := \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$. For the case $\xi = \zeta + 1$ for a countable ordinal $1 \leq \zeta < \omega_1$, we define \mathcal{S}_ξ as the set containing \emptyset together with all $F \subset \mathbb{N}$ such that there exist $n \in \mathbb{N}$ and a decomposition $F = \bigcup_{k=1}^n F_k$ with sets F_k from \mathcal{S}_ζ satisfying $n \leq F_1 < \dots < F_n$. In case ξ is a limit ordinal we fix a strictly increasing sequence (ζ_n) of non-limit-ordinals satisfying $\sup_n \zeta_n = \xi$, and define $\mathcal{S}_\xi := \bigcup_{n=1}^\infty \{F \in \mathcal{S}_{\zeta_n} : n \leq F\}$.

For convenience, in some contexts we will write \mathcal{S}_{ω_1} for the family of all finite subsets of \mathbb{N} . In other words, we have \mathcal{S}_{ω_1} being identical to $[\mathbb{N}]^{<\omega}$, and which notation we use will depend on the context. This is, admittedly, somewhat of an abuse of notation, as there is no such thing as the “ ω_1 st Schreier family,” but it will greatly simplify the writing.

It is well-known that the Schreier families (including \mathcal{S}_{ω_1}) have the **spreading property**, which is to say that if $\{n_1 < \dots < n_k\} \in \mathcal{S}_\xi$ and $\{m_1 < \dots < m_k\} \subseteq \mathbb{N}$ satisfies $n_j \leq m_j$ for all $j = 1, \dots, k$, then $\{m_1 < \dots < m_k\} \in \mathcal{S}_\xi$. Please note that, in general, it is not true that $\mathcal{S}_\xi \subseteq \mathcal{S}_\zeta$ for all $1 \leq \xi \leq \zeta \leq \omega_1$. However, we do always have $\mathcal{S}_1 \subseteq \mathcal{S}_\xi$, $1 \leq \xi \leq \omega_1$. Furthermore, given any $1 \leq \xi \leq \zeta \leq \omega_1$ there exist $d = d(\xi, \zeta)$ such that if $S \in \mathcal{S}_\xi$ with $d \leq \min S$ then $S \in \mathcal{S}_\zeta$. (That $\mathcal{S}_1 \subseteq \mathcal{S}_\xi$ is obvious, and the remaining facts can all be found, for instance, in [AGR03, III.2,p1051].)

We will also need a pair of technical results regarding the Schreier families.

1.1. Proposition ([Po09, Lemma 1.3]). *Let $1 \leq \xi \leq \omega_1$ and $A \in \mathcal{S}_\xi$. Then*

$$\{2n, 2n + 2 : n \in A\} \in \mathcal{S}_\xi.$$

1.2. Proposition ([OTW97, Proposition 3.2(b)]). *If $1 \leq \xi, \zeta < \omega_1$ then there is $L \in [\mathbb{N}]$ such that $\mathcal{S}_\xi[\mathcal{S}_\zeta](L) \subseteq \mathcal{S}_{\zeta+\xi}$.*

For a fixed ordinal $1 \leq \xi \leq \omega_1$, we say that a sequence (x_n) in a Banach space X **\mathcal{S}_ξ -dominates** another sequence (y_n) in a Banach space Y just in case there is a

constant $C \in [1, \infty)$ satisfying

$$\left\| \sum_{n \in F} a_n y_n \right\| \leq C \left\| \sum_{n \in F} a_n x_n \right\|$$

for all $(a_n) \in c_{00}$ and $F \in \mathcal{S}_\xi$. When $\xi = \omega_1$ we will simply say that (x_n) **dominates** (y_n) and this will coincide with the usual notion of domination in literature. In this case we write $(x_n) \geq_C (y_n)$ or, when C is unimportant, simply $(x_n) \geq (y_n)$.

The remainder of the paper is divided into two parts. In the next section we use Lorentz sequence spaces to show that class $\mathcal{WS}_{e,\xi}$, $1 \leq \xi \leq \omega_1$, fails to be closed under addition when $e = (e_n)$ is chosen from among the canonical bases for ℓ_p , $1 < p < \infty$. After that, in the last section, we define and study new classes $\mathcal{JS}_{e,\xi}$. There we show that, in the case when $e = (e_n)$ is the canonical basis for either ℓ_p , $1 \leq p < \infty$, or c_0 , $\mathcal{JS}_{e,\omega_1}$ is a norm-closed operator ideal. We conclude by using the new classes $\mathcal{JS}_{e,\xi}$ to show in Theorem 3.1 that $\mathcal{L}(X)$ admits infinitely many closed ideals whenever X contains complemented copies of ℓ_1 and either some ℓ_q , $1 < q \leq \infty$, or c_0 ; the same holds if X instead contains complemented copies of ℓ_∞ and some ℓ_p , $1 \leq p < \infty$.

2. CLASSES $\mathcal{WS}_{e,\xi}$

2.1. Definition. Let X and Y be Banach spaces, and let $1 \leq \xi \leq \omega_1$ be an ordinal. Fix any normalized basis $e = (e_n)$. We define $\mathcal{WS}_{e,\xi}(X, Y)$ as the set of all operators $T \in \mathcal{L}(X, Y)$ such that for any normalized basic sequence (x_n) in X , the image sequence (Tx_n) fails to \mathcal{S}_ξ -dominate (e_n) .

The following is a straightforward observation based on the basic constant of (e_n) .

2.2. Proposition. Let $1 \leq \xi \leq \omega_1$, let $e = (e_n)$ be any normalized basis, and let X and Y be Banach spaces with $T \in \mathcal{L}(X, Y)$. If (x_n) is a sequence in X and (Tx_n) has a norm-convergent subsequence, then for every $\epsilon > 0$ and $N > 0$ there exist m and n such that $N < n < m$ and

$$\|Tx_m - Tx_n\|_Y < \epsilon \|e_m - e_n\|_E.$$

If this happens for every normalized basic sequence in X , then $T \in \mathcal{WS}_{e,\xi}(X, Y)$. In particular, we always have $\mathcal{K}(X, Y) \subseteq \mathcal{WS}_{e,\xi}(X, Y)$.

In view of the last proposition, in several proofs, we will be concentrating on sequences with no convergent subsequences. Applying Rosenthal's ℓ_1 Theorem (cf., e.g., [AK06, Theorem 10.2.1]) together with [AK06, Theorem 1.5.4], we obtain the following standard fact.

2.3. Proposition. Suppose X and Y are Banach spaces with $T \in \mathcal{L}(X, Y)$. If (x_n) is a bounded sequence in X , then there exists a subsequence (x_{n_k}) so that exactly one of the following holds.

- (Tx_{n_k}) is norm-convergent.
- The sequences $(x_{n_{4k+2}} - x_{n_{4k}})$ and $(Tx_{n_{4k+2}} - Tx_{n_{4k}})$ are both seminormalized and basic, and each of them are either weakly null or else equivalent to the canonical basis of ℓ_1 .

This gives us an equivalent characterization of the classes in terms of bounded sequences instead of normalized basic sequences.

2.4. Proposition. Let X and Y be Banach spaces, and let $1 \leq \xi \leq \omega_1$ be an ordinal. Fix any normalized basis $e = (e_n)$ satisfying $(e_{4n+2} - e_{4n}) \geq (e_n)$. Then $\mathcal{WS}_{e,\xi}(X, Y)$

is the set of all operators $T \in \mathcal{L}(X, Y)$ such that for any bounded sequence (x_n) in X , the image sequence (Tx_n) fails to \mathcal{S}_ξ -dominate (e_n) .

Proof. Suppose T is in $\mathcal{WS}_{e,\xi}(X, Y)$ and fix any bounded sequence (x_n) . If (Tx_n) has a norm-convergent subsequence then we are done by Proposition 2.2. Otherwise, by Proposition 2.3 we can find a subsequence (x_{n_k}) so that (z_k) and (Tx_k) are each seminormalized basic, where we define

$$z_k := x_{n_{4k+2}} - x_{n_{4k}}.$$

Fix $\epsilon > 0$, and pass to a further subsequence if necessary so that $\|Tx_k\|_Y \rightarrow r$ for some $0 < r < \infty$, and quickly enough so that by the Principle of Small Perturbations, $(\frac{1}{r}e_k)$ is C -equivalent, $C \geq 1$, to $(\frac{1}{\|z_k\|_X}e_k)$. Notice that $(\frac{z_k}{\|z_k\|_X})$ is a normalized basic sequence in X . Since $T \in \mathcal{WS}_{e,\xi}(X, Y)$, we can therefore find $(a_n) \in c_{00}$ with support in \mathcal{S}_ξ such that

$$\left\| \sum a_k T \frac{z_k}{\|z_k\|_X} \right\|_Y < \frac{\epsilon}{Cr} \left\| \sum a_k e_k \right\|_E \leq \epsilon \left\| \sum \frac{a_k}{\|z_k\|_X} e_k \right\|_E.$$

This is sufficient by Proposition 1.1 together with the spreading property of \mathcal{S}_ξ and fact that $(e_n) \leq (e_{4n+2} - e_{4n})$. ■

For the remainder of this section, let us develop the machinery required to show that class $\mathcal{WS}_{e,\xi}$ fails to be closed under addition whenever (e_n) is the canonical basis for ℓ_p , $1 < p < \infty$.

2.5. Proposition. *Suppose (e_n) is a normalized basic sequence satisfying $(e_{2n+2} - e_{2n}) \geq (e_n)$, and fix $1 \leq \xi \leq \omega_1$. Let X_1 and X_2 be Banach spaces, T be an operator from $\mathcal{WS}_{e,\xi}(X_1, X_2)$, and let Y_1 and Y_2 denote two more Banach spaces. Then*

$$T \oplus 0 \in \mathcal{WS}_{e,\xi}(X_1 \oplus Y_1, X_2 \oplus Y_2) \quad \text{and} \quad 0 \oplus T \in \mathcal{WS}_{e,\xi}(Y_1 \oplus X_1, Y_2 \oplus X_2).$$

Proof. By symmetry it suffices to prove the first statement. Let $(x_n \oplus y_n)$ be a bounded sequence in $X_1 \oplus Y_1$, and let $\epsilon > 0$. Then (x_n) is bounded in X_1 , and so by Proposition 2.4 we can find $(a_k) \in c_{00}$ with support in \mathcal{S}_ξ such that

$$\left\| \sum a_n (T \oplus 0)(x_n \oplus y_n) \right\|_{X_2 \oplus Y_2} = \left\| \sum a_n T x_n \right\|_{X_2} < \epsilon \left\| \sum a_n e_n \right\|_E.$$

■

Let us now describe the Lorentz sequence spaces. Suppose $1 \leq q < \infty$, and let $w = (w_n)_{n=1}^\infty \in c_0 \setminus \ell_1$ be a nonincreasing sequence with $w_1 = 1$. Denote by Π the set of all permutations of \mathbb{Z}^+ . We define the set $d(w, q)$ as the set of all scalar sequences $(a_n)_{n=1}^\infty$ such that

$$\|(a_n)_{n=1}^\infty\|_{d(w,q)} := \sup_{\sigma \in \Pi} \left(\sum_{n=1}^\infty |a_{\sigma(n)}|^q w_n \right)^{\frac{1}{q}} < \infty.$$

When endowed with the norm $\|\cdot\|_{d(w,q)}$, the set $d(w, q)$ defines a Banach space with a canonical basis which is normalized and symmetric. Note that, due to the properties of w , we can equivalently characterize the $d(w, q)$ norm as follows. If $(a_n)_{n=1}^\infty \in d(w, q)$, then we denote by $(a_n^*)_{n=1}^\infty$ the nonincreasing rearrangement of $(|a_n|)_{n=1}^\infty$. In this case,

$$\|(a_n)_{n=1}^\infty\|_{d(w,q)} = \|(a_n^* w_n^{1/q})_{n=1}^\infty\|_{\ell_q}.$$

We shall call any such space $d(w, q)$ a **Lorentz sequence space**. See [LT77, §4.e] for further discussion of these spaces.

2.6. Proposition. Fix numbers p and q such that $1 \leq q < p < \infty$. Let $w = (w_n)$ and $\tilde{w} = (\tilde{w}_n)$ be nonincreasing sequences, $w_1 = \tilde{w}_1 = 1$, lying in $c_0 \setminus \ell_1$, with (x_n) and (\tilde{x}_n) the respective canonical bases for $d(w, q)$ and $d(\tilde{w}, q)$. Suppose that $w_n + \tilde{w}_n \geq n^{\frac{1}{p} - \frac{1}{q}}$. Then $(x_n \oplus \tilde{x}_n) \subset X$ 1-dominates the canonical basis of ℓ_p , where we define

$$X := (d(w, q) \oplus d(\tilde{w}, q))_{\ell_q}.$$

Proof. Set, for convenience, $v_n := w_n + \tilde{w}_n$. Our goal is to prove that for each k and every $(a_n) \in c_{00}$ with $\text{supp}(a_n) \subset [1, k]$ we have the following inequality:

$$\left\| \sum a_n (x_n \oplus \tilde{x}_n) \right\|_X^q = \left\| \sum_{n=1}^k a_n (x_n \oplus \tilde{x}_n) \right\|_X^q = \sum_{n=1}^k a_n^{*q} v_n \geq \|(a_n)\|_{\ell_p}^q.$$

Clearly, the result holds when $k = 1$. Now assume it holds for some $k \in \mathbb{N}$ and let us prove the inequality for $k + 1$. Using the inductive assumption, we can write

$$\sum_{n=1}^{k+1} a_n^{*q} v_n \geq \|(a_n^*)_{n=1}^k\|_{\ell_p}^q + a_{k+1}^{*q} v_{k+1} \geq \|(a_n^*)_{n=1}^k\|_{\ell_p}^q + a_{k+1}^{*q} (k+1)^{\frac{1}{p} - \frac{1}{q}}$$

Define a function $f : [0, a_k^*] \rightarrow \mathbb{R}$ by the rule

$$f(t) = \|(a_n^*)_{n=1}^k\|_{\ell_p}^q + t^q (k+1)^{\frac{1}{p} - \frac{1}{q}} - \left(\sum_{n=1}^k a_n^{*p} + t^p \right)^{\frac{q}{p}}.$$

Notice that if we show that f is nonnegative, then we are done. For that, observe

$$\begin{aligned} f'(t) &= qt^{q-1} (k+1)^{\frac{1}{p} - \frac{1}{q}} - qt^{p-1} \left(\sum_{n=1}^k a_n^{*p} + t^p \right)^{\frac{q}{p} - 1} \\ &= qt^{q-1} \left[(k+1)^{\frac{1}{p} - \frac{1}{q}} - t^{p-q} \left(\sum_{n=1}^k a_n^{*p} + t^p \right)^{\frac{q-p}{p}} \right] \quad \text{then, as each } a_n^* \geq a_k^* \geq t, \\ &\geq qt^{q-1} \left[(k+1)^{\frac{1}{p} - \frac{1}{q}} - t^{p-q} ((k+1)t^p)^{\frac{q-p}{p}} \right] = qt^{q-1} \left[(k+1)^{\frac{q-p}{pq}} - (k+1)^{\frac{q-p}{p}} \right]. \end{aligned}$$

The latter is nonnegative for each $t \in [0, a_k^*]$ due to $1 \leq q < p$. Since $f(0) = 0$ the proposition is proved. ■

2.7. Proposition. Let $1 \leq q < p < \infty$. There exists a pair of nonincreasing sequences $w = (w_n)$ and $\tilde{w} = (\tilde{w}_n)$, $w_1 = \tilde{w}_1 = 1$, lying in $c_0 \setminus \ell_1$, such that neither of the respective canonical bases (x_n) or (\tilde{x}_n) of $d(w, q)$ and $d(\tilde{w}, q)$ dominates the canonical basis of ℓ_p , but their ℓ_q -direct sum $(x_n \oplus \tilde{x}_n)$ does.

Proof. Put $s := \frac{1}{q} - \frac{1}{p}$, and as usual, let $(s_n) \subset c_{00}$ denote the summing basis, i.e. the basis defined by

$$s_n = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, 0, 0, 0, \dots).$$

We will construct sequences w and \tilde{w} together with a sequence of indices

$$0 = m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \dots$$

inductively. For each n we will have either $w_n = \frac{1}{n^s}$ or else $\tilde{w}_n = \frac{1}{n^s}$. By Proposition 2.6, this will guarantee that $(x_n \oplus \tilde{x}_n)$ dominates the canonical basis of ℓ_p . Second, we will have $\|s_{n_k}\|_{d(\tilde{w}, q)} \leq k^{-1} \|s_{n_k}\|_{\ell_p}$ if k is odd and $\|s_{n_k}\|_{d(w, q)} \leq k^{-1} \|s_{n_k}\|_{\ell_p}$ otherwise. Thus, neither (x_n) nor (\tilde{x}_n) will dominate the canonical basis of ℓ_p .

Begin by defining $n_1 = 1$ and $w_1 = \tilde{w}_1 = 1$. Notice that this means $\|s_{n_1}\|_{d(\tilde{w},q)} \leq \|s_{n_1}\|_{\ell_p}$ and $w_{n_1} = n_1^{-s}$.

For the inductive step, suppose $m_1 < n_1 < \dots < m_k < n_k$, $(w_n)_{n=1}^{n_k}$, and $(\tilde{w}_n)_{n=1}^{n_k}$ have all been defined, and consider the case where $k+1$ is even. Assume, in addition, that $w_{n_k} = n_k^{-s}$. Set $m_{k+1} > n_k$ to be any number such that $\tilde{w}_{n_k} \geq (m_{k+1} + 1)^{-s}$. Define $w_j = \frac{1}{j^s}$ and $\tilde{w}_j = \tilde{w}_{n_k}$ for all $n_k < j \leq m_{k+1}$. Due to the inductive hypothesis, we have $w_{n_{k+1}} < n_k^{-s} \leq w_{n_k}$ so that both sequences are nonincreasing as required. Also, by the choice of m_{k+1} , we have $\tilde{w}_{m_{k+1}} \geq (m_{k+1} + 1)^{-s}$.

Next, let $n_{k+1} > m_{k+1}$ be such that

$$\|s_{m_k}\|_{d(w,q)} + 1 \leq \frac{1}{k+1} \|s_{n_{k+1}}\|_{\ell_p}.$$

Define $\tilde{w}_j = \frac{1}{j^s}$ and $w_j = \frac{1}{2^j} w_{m_{k+1}}$ for all $m_{k+1} < j \leq n_{k+1}$. Then

$$\|s_{n_{k+1}}\|_{d(w,q)} \leq \|s_{m_{k+1}}\|_{d(w,q)} + \left(\sum_{j \geq 1} \frac{w_{m_{k+1}}}{2^j} \right)^{\frac{1}{q}} \leq \|s_{m_{k+1}}\|_{d(w,q)} + 1 \leq \frac{\|s_{n_{k+1}}\|_{\ell_p}}{k+1}.$$

The case where $k+1$ is odd we handle in a similar fashion. ■

We also need the following result from [KPSTT12].

2.8. Proposition ([KPSTT12], Lemma 4.10). *Let $1 \leq q < \infty$ and $w = (w_n) \in c_0 \setminus \ell_1$ be nonincreasing with $w_1 = 1$. Let $I_{q,w} : \ell_q \rightarrow d(w,q)$ denote the formal identity between canonical bases. Suppose (x_n) is a seminormalized block basic sequence in ℓ_q . If $(I_{q,w}x_n)$ is seminormalized in $d(w,q)$ then it admits a subsequence equivalent to the canonical basis of $d(w,q)$.*

2.9. Proposition. *Let $1 < q < p < \infty$ and $1 \leq \xi \leq \omega_1$, and suppose (e_n) is the canonical basis of ℓ_p . Let $d(w,q)$ be a Lorentz sequence space whose canonical basis fails to dominate the canonical basis of ℓ_p . Then the formal identity $I_{q,w} : \ell_q \rightarrow d(w,q)$ is class $\mathcal{WS}_{e,\xi}$.*

Proof. Fix any $\epsilon > 0$ and consider a normalized basic sequence $(x_n)_{n=1}^\infty$ in ℓ_q . Since every seminormalized basic sequence in a reflexive space is weakly null (as being shrinking), by the Bessaga-Pelczyński Selection Principle, we can find a subsequence (x_{n_k}) and successive finite subsets of \mathbb{N} which we denote

$$E_1 < E_2 < E_3 < \dots$$

so that $(E_k x_{n_k})$ is a seminormalized block basic sequence in ℓ_q , and

$$z_k := x_{n_k} - E_k x_{n_k}$$

satisfies $\|z_k\|_{\ell_q} \rightarrow 0$. By Proposition 2.2, we may assume that the $d(w,q)$ -block sequence $(I_{q,w} E_k x_{n_k})_{k=1}^\infty$ is seminormalized. Thus, we can pass to a further subsequence if necessary so that by Proposition 2.8, $(I_{q,w} E_k x_{n_k})_{k=1}^\infty$ is K -equivalent to the canonical basis of $d(w,q)$, where $K \geq 1$. Pass to a still further subsequence so that $\|I_{q,w} z_k\|_{d(w,q)} \leq 2^{-k-1} \epsilon$. Now, since the canonical basis of $d(w,q)$ fails to dominate the canonical basis of ℓ_p , we can find $(c_n) \in c_{00}$ such that

$$\|(c_k)\|_{d(w,q)} < \frac{\epsilon}{2K} \|(c_k)\|_{\ell_p}.$$

By the symmetric property of the canonical basis of $d(w, q)$ we can assume $\text{supp}(c_k) \in \mathcal{S}_1 \subseteq \mathcal{S}_\xi$. Then

$$\begin{aligned} \left\| \sum c_k I_{q,w} x_{n_k} \right\|_{d(w,q)} &\leq \left\| \sum c_k I_{q,w} E_k x_{n_k} \right\|_{d(w,q)} + \sum \|c_k I_{q,w} z_k\|_{d(w,q)} \\ &\leq K \|(c_k)\|_{d(w,q)} + \frac{\epsilon}{2} \|(c_k)\|_{\ell_\infty} < \frac{\epsilon}{2} \|(c_k)\|_{\ell_p} + \frac{\epsilon}{2} \|(c_k)\|_{\ell_\infty} \leq \epsilon \|(c_k)\|_{\ell_p}. \end{aligned}$$

■

We are now ready to prove the main result of this section.

2.10. Theorem. *Let $e = (e_n)$ denote the canonical basis of ℓ_p , $1 < p < \infty$, and let $1 \leq \xi \leq \omega_1$ be an ordinal. Then class $\mathcal{WS}_{e,\xi}$ fails to be closed under addition, and hence is not an operator ideal.*

Proof. Let $q \in (1, p)$ and choose w and \tilde{w} as in Proposition 2.7 so that neither of the respective canonical bases (x_n) or (\tilde{x}_n) of $d(w, q)$ and $d(\tilde{w}, q)$ dominates the canonical basis of ℓ_q , but their ℓ_q -direct sum $(x_n \oplus \tilde{x}_n)$ does. Let $I_{q,w} : \ell_q \rightarrow d(w, q)$ and $I_{q,\tilde{w}} : \ell_q \rightarrow d(\tilde{w}, q)$ denote the formal identity operators. Then by Propositions 2.5 and 2.9,

$$I_{q,w} \oplus 0 : \ell_q \oplus \ell_q \rightarrow d(w, q) \oplus d(\tilde{w}, q) \quad \text{and} \quad 0 \oplus I_{q,\tilde{w}} : \ell_q \oplus \ell_q \rightarrow d(w, q) \oplus d(\tilde{w}, q)$$

are both class $\mathcal{WS}_{e,\xi}$. However, since $(x_n \oplus \tilde{x}_n)$ dominates the canonical basis of ℓ_p , their sum

$$I_{q,w} \oplus 0 + 0 \oplus I_{q,\tilde{w}} = I_{q,w} \oplus I_{q,\tilde{w}}$$

is not class $\mathcal{WS}_{e,\xi}$. ■

3. CLOSED IDEALS IN $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q < \infty$, AND $\mathcal{L}(\ell_1 \oplus c_0)$

For many years, researchers have been interested in discovering whether or not, given a particular Banach space X , the operator algebra $\mathcal{L}(X)$ admits infinitely many closed ideals. In the case of many classical Banach spaces, this has long been decided. For instance, in 1960 it was shown that $\mathcal{L}(\ell_p)$, $1 \leq p < \infty$, and $\mathcal{L}(c_0)$ admit exactly three closed ideals ([GMF67]). This also took care of the case $\mathcal{L}(L_2)$, since $L_2 \cong \ell_2$. By 1978 it was discovered that $\mathcal{L}(L_p)$ admits infinitely many closed ideals for $p \in (1, 2) \cup (2, \infty)$ ([Pi78, Theorem 5.3.9]), and in 2015 this was improved to show continuum many ([Wa15, Theorem 1.1]). Also in 1978 was shown that $\mathcal{L}(C[0, 1])$ admits uncountably many closed ideals ([Pi78, Theorem 5.3.11]). Whether $\mathcal{L}(L_1)$ and $\mathcal{L}(L_\infty) \cong \mathcal{L}(\ell_\infty)$ admit infinitely many closed ideals remains a significant open question.

Besides these classical cases, the closed ideal structures of $\mathcal{L}(\ell_p \oplus \ell_q)$, $1 \leq p < q < \infty$, have generated a great deal of interest. Although Pietsch asked as early as 1978 whether these operator algebras admit infinitely many closed ideals ([Pi78, Problem 5.33]), the question remained entirely open for over 36 years. Indeed, not until 2014 was it finally shown that $\mathcal{L}(\ell_p \oplus \ell_q)$ admits continuum many closed ideals whenever $1 < p < q < \infty$ ([SZ14]). Then, in 2015 was shown that this result extends to $\mathcal{L}(\ell_p \oplus c_0)$ and $\mathcal{L}(\ell_1 \oplus \ell_q)$ in the special cases $1 < p < 2 < q < \infty$ ([Wa15, Theorem 1.1]).

In this section we close Pietsch's question by proving the following.

3.1. Theorem. *Suppose X is a Banach space containing a complemented copy of ℓ_1 , and a complemented copy either of ℓ_q , $1 < q \leq \infty$, or of c_0 . Then $\mathcal{L}(X)$ admits an uncountable chain of closed ideals. The same is true if X contains a complemented*

copy of ℓ_p , $1 \leq p < \infty$, and of ℓ_∞ . In particular, $\mathcal{L}(\ell_1 \oplus \ell_q)$, $1 < q \leq \infty$, and $\mathcal{L}(\ell_1 \oplus c_0)$ each admit an uncountable chain of closed ideals.

Unfortunately, the cases $\mathcal{L}(\ell_p \oplus c_0)$ fail to dualize, and remain open for $2 \leq p < \infty$.

Note that in addition to the above operator algebras, we can also close some of the remaining cases for Rosenthal's X_p spaces and Woo's $X_{p,r}$ generalizations thereof. Let us take a moment to recall the definitions of these spaces. Pick any $1 \leq r < p \leq \infty$. Let (f_n) and (g_n) denote the respective canonical bases of ℓ_p (or c_0 if $p = \infty$) and ℓ_r . Let $(w_n) \in c_0$ be any sequence of positive numbers tending to zero, and satisfying the condition that $\sum w_n^{pr/(p-r)} = \infty$ (or $\sum w_n^r = \infty$ if $p = \infty$). Set $e_n = f_n \oplus_\infty w_n g_n$, vectors lying in $\ell_p \oplus_\infty \ell_r$ (or $c_0 \oplus_\infty \ell_r$ if $p = \infty$). Then we can define Woo's spaces $X_{p,r} = [e_n]$. Rosenthal's spaces are just special cases of Woo's spaces, and we may define them for any $p \in [1, 2) \cup (2, \infty]$ by setting $X_p = X_{p,2}$ if $p \in (2, \infty]$ and $X_p = X_{p',2}^*$, $\frac{1}{p} + \frac{1}{p'} = 1$, if $p \in [1, 2)$. It had previously been observed that by [Wa15, Theorem 1.1] the operator algebra $\mathcal{L}(X_{p,r})$ admits continuum many closed ideals whenever $1 < r < p < \infty$, or whenever $p = \infty$ and $r \in (1, 2)$. Indeed, so do their dual space algebras $\mathcal{L}(X_{p,r}^*)$, for the same choices of r and p . Note that this also gives continuum many closed ideals in $\mathcal{L}(X_p)$ and $\mathcal{L}(X_p^*)$ whenever $p \in (1, 2) \cup (2, \infty)$. However, due to the fact that $X_{p,r}$ always contains complemented copies of ℓ_p (or c_0 , if $p = \infty$) and ℓ_r (cf. [Woo75, Corollary 3.2]), by Theorem 3.1 we now have uncountably many closed ideals in $\mathcal{L}(X_{p,1})$ and $\mathcal{L}(X_{p,1}^*)$ for all choices of $1 < p \leq \infty$, and $\mathcal{L}(X_{\infty,r}^*)$ for all $1 \leq r < \infty$. So do $\mathcal{L}(X_{\infty,1})$, $\mathcal{L}(X_1)$, $\mathcal{L}(X_1^*)$, and $\mathcal{L}(X_\infty^*)$. Among Woo's and Rosenthal's spaces and their duals, this leaves open only the cases $\mathcal{L}(X_{\infty,r})$, $r \in [2, \infty)$, and $\mathcal{L}(X_\infty)$.

We will prove Theorem 3.1 by modifying the definition of classes $\mathcal{WS}_{e,\xi}$, and using them to produce uncountable chains of closed subideals in certain operator algebras. The new classes are as follows.

3.2. Definition. Let X and Y be Banach spaces, and let $1 \leq \xi \leq \omega_1$ be an ordinal. Fix any normalized basis $e = (e_n)$. We define $\mathcal{JS}_{e,\xi}(X, Y)$ as the set of all operators $T \in \mathcal{L}(X, Y)$ such that for any normalized basic sequence (x_n) in X satisfying $(x_n) \leq (e_n)$, the image sequence (Tx_n) fails to \mathcal{S}_ξ -dominate (e_n) .

So, we have weakened the definition of class $\mathcal{WS}_{e,\xi}$ by considering only those normalized basic sequences which are dominated by (e_n) . This will ensure that we can get closure under addition in the non-Schreier cases, that is, for classes $\mathcal{JS}_{e,\omega_1}$.

Note that due to the strength of the ℓ_1 norm, every normalized basic sequence is dominated by the canonical basis of ℓ_1 , and so in case $e = (e_n)$ is the canonical basis for ℓ_1 we have

$$\mathcal{R}_\xi = \mathfrak{SM}_1^\xi = \mathcal{WS}_{e,\xi} = \mathcal{JS}_{e,\xi}, \quad 1 \leq \xi \leq \omega_1.$$

Here, $\mathcal{R}_{\omega_1} = \mathcal{R}$ denotes the Rosenthal operators, and $\mathcal{R}_\xi = \mathcal{WS}_{e,\xi}$, $1 \leq \xi < \omega_1$, denotes the ξ th-order *Schreier Rosenthal* operators, defined in [BF11]; classes \mathfrak{SM}_1^ξ , $1 \leq \xi \leq \omega_1$, were defined in [BCFW15, §4]. Hence, each of these forms a norm-closed operator ideal by [BCFW15, Theorem 4.3]. However, if $e = (e_n)$ is the canonical basis for ℓ_p , $1 < p < \infty$, or c_0 , then we do not yet know whether $\mathcal{JS}_{e,\xi}$ is closed under addition for any $1 \leq \xi < \omega_1$.

Let us now observe some straightforward consequences of the definition of $\mathcal{JS}_{e,\xi}$.

3.3. Proposition. Let X and Y be Banach spaces, let $1 \leq \xi \leq \omega_1$, and let $e = (e_n)$ be the canonical basis for ℓ_p , $1 \leq p < \infty$, or c_0 .

- (1) An operator $T \in \mathcal{L}(X, Y)$ is class $\mathcal{JS}_{e,\xi}$ just in case for every bounded sequence (x_n) in X which is dominated by (e_n) , and every $\epsilon > 0$, there exists $(a_n) \in c_{00}$ and $F \in \mathcal{S}_\xi$ such that $\|\sum a_n T x_n\| < \epsilon \|\sum a_n e_n\|$.

(2) If ζ is another ordinal with $1 \leq \xi \leq \zeta \leq \omega_1$, then

$$\mathcal{JS}_{e,\xi}(X, Y) \subseteq \mathcal{JS}_{e,\zeta}(X, Y).$$

(3) Every compact operator in $\mathcal{L}(X, Y)$ is class $\mathcal{JS}_{e,\xi}$. In other words,

$$\mathcal{K}(X, Y) \subseteq \mathcal{JS}_{e,\xi}(X, Y).$$

(4) Suppose Z is also a Banach space, $T \in \mathcal{L}(X, Y)$ is an operator, and $J : Y \rightarrow Z$ is a continuous linear embedding. If $T \notin \mathcal{JS}_{e,\xi}(X, Y)$ then $JT \notin \mathcal{JS}_{e,\xi}(X, Z)$.

(5) $\mathcal{JS}_{e,\xi}(X, Y)$ is a norm-closed subset of $\mathcal{L}(X, Y)$.

(6) Suppose W and Z are also Banach spaces. If $T \in \mathcal{JS}_{e,\xi}(X, Y)$, $A \in \mathcal{L}(W, X)$, and $B \in \mathcal{L}(Y, Z)$, then $BT A \in \mathcal{JS}_{e,\xi}(W, Z)$.

Proof. The proof of (1) is almost identical to the proof of Proposition 2.4, so we omit it.

(2) follows from the spreading property for Schreier families, together with the fact that for any pair of ordinals $1 \leq \xi \leq \zeta \leq \omega_1$ there exists $d = d(\xi, \zeta)$ such that for any $S \in \mathcal{S}_\xi$ with $d \leq \min S$ we have $S \in \mathcal{S}_\zeta$.

(3) is just another consequence of Proposition 2.2.

(4) is immediate from the definition of $\mathcal{JS}_{e,\xi}$.

Let us now prove (5). Suppose (T_j) is a sequence in $\mathcal{JS}_{e,\xi}(X, Y)$ with $T_j \rightarrow T$ for some $T \in \mathcal{L}(X, Y)$. Let (x_n) be a normalized basic sequence in X which is C -dominated, $C \in [1, \infty)$, by (e_n) , and let $\epsilon > 0$. Find T_j with $\|T_j - T\| < \frac{\epsilon}{2C}$. Now let $(a_n)_{n \in F}$, $F \in \mathcal{S}_\xi$ be such that

$$\left\| \sum_{n \in F} a_n T_j x_n \right\| < \frac{\epsilon}{2} \left\| \sum_{n \in F} a_n e_n \right\|.$$

Then

$$\left\| \sum_{n \in F} a_n T x_n \right\| \leq \left\| \sum_{n \in F} a_n T_j x_n \right\| + \|T - T_j\| \left\| \sum_{n \in F} a_n x_n \right\| < \epsilon \left\| \sum_{n \in F} a_n e_n \right\|.$$

This completes the proof of (5).

Lastly, we shall prove (6). Let (w_n) be a bounded sequence in W which is dominated by (e_n) , and let $\epsilon > 0$. Then (Aw_n) is a bounded sequence in X which is dominated by (e_n) , which means by (1) that we can find $F \in \mathcal{S}_\xi$ and $(a_n)_{n \in F}$ so that

$$\left\| \sum_{n \in F} a_n B T A w_n \right\| \leq \|B\| \left\| \sum_{n \in F} T A w_n \right\| < \frac{\epsilon}{\|A\|} \left\| \sum_{n \in F} A w_n \right\| \leq \epsilon \left\| \sum_{n \in F} A w_n \right\|.$$

■

3.4. Remark. Observe that in the previous Proposition 3.3, the assumption $e = (e_n)$ is the canonical basis for ℓ_p , $1 \leq p < \infty$, or c_0 , is only required for parts (1) and (6). In parts (2), (3), (4), and (5), $e = (e_n)$ could be any normalized basis for a Banach space E .

If we want classes $\mathcal{JS}_{e,\xi}$ form operator ideals, it remains to show that they are closed under addition. In case $1 \leq \xi < \omega_1$, this is unknown. However, below we present a partial result in that direction, which turns out to be sufficient for our purposes here.

3.5. Proposition. Let X and Y be Banach spaces, let $e = (e_n)$ denote the canonical basis for ℓ_p , $1 \leq p < \infty$, or c_0 , and let $1 \leq \xi, \zeta \leq \omega_1$ be ordinals. Suppose $S \in \mathcal{JS}_{e,\xi}(X, Y)$ and $T \in \mathcal{JS}_{e,\zeta}(X, Y)$.

- (i) If $\xi = \omega_1$ or $\zeta = \omega_1$ then $S + T \in \mathcal{JS}_{e, \omega_1}(X, Y)$.
- (ii) If ξ and ζ are both countable, i.e. $< \omega_1$, then $S + T \in \mathcal{JS}_{e, \xi + \zeta}(X, Y)$.

Proof. Pick any bounded sequence (x_n) in X which is dominated by (e_n) . By Proposition 1.2 we can find $L = (n_k) \in [\mathbb{N}]$ such that $\mathcal{S}_\xi[\mathcal{S}_\zeta](L) \subseteq \mathcal{S}_{\zeta + \xi}$ in case (ii), and let $L = (n_k) = \mathbb{N}$ in case (i). Now, by successively considering the tails of (x_{n_k}) and using the spreading property of \mathcal{S}_ζ we can find $F_1 < F_2 < F_3 < \dots \in \mathcal{S}_\zeta$ and scalars (a_k) such that

$$\left\| \sum_{k \in F_j} a_k T x_{n_k} \right\| < \epsilon 2^{-j-1} \quad \text{and} \quad \left\| \sum_{k \in F_j} a_k e_k \right\| = 1 \quad \text{for all } j \in \mathbb{N}.$$

Let us form matching block sequences by setting

$$x'_j = \sum_{k \in F_j} a_k x_{n_k}, \quad \text{and} \quad e'_j = \sum_{k \in F_j} a_k e_k, \quad j \in \mathbb{N}.$$

Recall that every normalized block sequence of (e_n) is 1-equivalent to (e_n) (cf., e.g., [AK06, Lemma 2.1.1]). In particular, (e'_j) is 1-equivalent to (e_n) , which means (x'_j) is dominated by (e_n) . We can therefore find scalars $(b_j) \in c_{00}$ and $F \in \mathcal{S}_\xi$ such that

$$\left\| \sum_{j \in F} b_j S x'_j \right\| < \frac{\epsilon}{2} \quad \text{and} \quad \left\| \sum_{j \in F} b_j e'_j \right\| = \left\| \sum_{j \in F} b_j e_j \right\| = 1.$$

Next, due to Hölder's inequality, we obtain

$$\left\| \sum_{j \in F} b_j (S + T) x'_j \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \sum_{j \in F} |b_j| 2^{-j} \leq \epsilon = \epsilon \left\| \sum_{j \in F} b_j e'_j \right\|.$$

In case (i) we are already done, and in case (ii) we need only recall that $\mathcal{S}_\xi[\mathcal{S}_\zeta](L) \subseteq \mathcal{S}_{\zeta + \xi}$, so that we are done anyway. ■

The limitations on the above proposition prevent us from concluding that $\mathcal{JS}_{e, \xi}$ is an operator ideal when $1 \leq \xi < \omega_1$. However, if we combine Proposition 3.3(3),(5),(6), and Proposition 3.5, we obtain the following nice result when $\xi = \omega_1$.

3.6. Theorem. *Let $e = (e_n)$ denote the canonical basis for ℓ_p , $1 \leq p < \infty$, or c_0 . Then $\mathcal{JS}_{e, \omega_1}$ is a norm-closed operator ideal.*

We also have the following property for this same special case $\xi = \omega_1$.

3.7. Proposition. *Let $e = (e_n)$ be a normalized basis for a Banach space E , and let X and Y be Banach spaces such that either X or Y fails to contain a copy of E . Then*

$$\mathcal{JS}_{e, \omega_1}(X, Y) = \mathcal{L}(X, Y).$$

Proof. Indeed, suppose $\mathcal{JS}_{e, \xi}(X, Y) \neq \mathcal{L}(X, Y)$. Then there is a linear operator $T \in \mathcal{L}(X, Y) \setminus \mathcal{JS}_{e, \xi}(X, Y)$, which means we can find a normalized basic sequence (x_n) in X which is dominated by (e_n) , and such that (Tx_n) dominates (e_n) . Then

$$(x_n) \leq (e_n) \leq (Tx_n) \leq (x_n)$$

so that X and Y both contain copies of E . ■

Recall that every seminormalized basic sequence in a reflexive space is weakly null. Together with the Bessaga-Pelczyński Selection Principle and the Principle of Small Perturbations, this means that any seminormalized basic sequence in a reflexive space with a basis admits a subsequence equivalent to a normalized block basic

sequence. In the case of ℓ_q , $1 < q < \infty$, this is in turn equivalent to the canonical basis (cf., e.g., [AK06, Lemma 2.1.1]), so that every normalized basic sequence has a subsequence dominated by the canonical basis (e_n) of ℓ_p , $1 \leq p \leq q$. Obviously, if $q = 1$ and $1 \leq p \leq q$ then $p = 1$ so that, again, every normalized basic sequence in ℓ_q is dominated by (e_n) . From this we obtain the following.

3.8. Proposition. *Let Y be a Banach space, let $1 \leq \xi \leq \omega_1$ be an ordinal, and let $e = (e_n)$ denote the canonical basis for ℓ_p , where $1 \leq p \leq q < \infty$. Then*

$$\mathcal{JS}_{e,\xi}(\ell_q, Y) = \mathcal{WS}_{e,\xi}(\ell_q, Y).$$

Consequently, in these cases we can apply a nice result from [BF11].

3.9. Theorem ([BF11, Corollary 19]). *Let X and Y be separable Banach spaces, and let $e = (e_n)$ denote any normalized 1-spreading basis. (In particular, $e = (e_n)$ can be chosen from among the canonical bases for ℓ_p , $1 \leq p < \infty$, or c_0 .) If $T \in \mathcal{WS}_{e,\omega_1}(X, Y)$ then there exists a countable ordinal $1 \leq \xi < \omega_1$ such that $T \in \mathcal{WS}_{e,\xi}(X, Y)$.*

This shall be used to prove the following.

3.10. Theorem. *Let $1 \leq p < \infty$, and let $Z = \ell_{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$ if $p \neq 1$ and $Z = c_0$ if $p = 1$. Then $\mathcal{L}(\ell_1, Z)$ and $\mathcal{L}(\ell_p, \ell_\infty)$ each admit an uncountable chain of strictly increasing closed subideals.*

Proof. Denote by $e = (e_n)$ the canonical basis for ℓ_p . For a countable ordinal $1 \leq \xi < \omega_1$, let us set

$$\mathcal{J}_\xi = \overline{\bigcup_{k=1}^{\infty} \mathcal{JS}_{e,\xi k}(\ell_p, \ell_\infty)}.$$

It is clear from Proposition 3.3(2),(5),(6), and Proposition 3.5, that if $\zeta := \sup_k \xi k$ then \mathcal{J}_ξ is a closed subideal of $\mathcal{L}(\ell_p, \ell_\infty)$ contained in $\mathcal{JS}_{e,\zeta}(\ell_p, \ell_\infty)$. Hence

$$\mathcal{J}_\xi^* = \{T \in \mathcal{L}(\ell_1, Z) : T^* \in \mathcal{J}_\xi\}.$$

is a closed subideal of $\mathcal{L}(\ell_1, Z)$.

We claim that there is a countable ordinal $\alpha > \zeta$ and an operator $A_\alpha \in \mathcal{L}(\ell_1, Z)$ such that $A_\alpha^* \in \mathcal{J}_\alpha \setminus \mathcal{J}_\zeta$. For proof of the claim, let T_ζ^p denote the p -convexification of the Tsirelson space of order ζ . It is well-known that T_ζ^p is a reflexive space containing no copy of ℓ_p , with a normalized basis (t_n) which is dominated by (e_n) , and which \mathcal{S}_ζ -dominates (e_n) (cf., e.g., [Wa14, §3, p88]). Thus, there exists an operator $I_\zeta \in \mathcal{L}(T_\zeta^{p*}, Z)$ whose dual is the continuous formal inclusion of $j_\zeta : \ell_p \rightarrow T_\zeta^p$ which maps $e_n \mapsto t_n$. Indeed, in the case $1 < p < \infty$, due to reflexivity of both spaces we can simply set $I_\zeta = (j_\zeta)^*$. In case $p = 1$, due to the weakness of the c_0 norm we can straightforwardly define $I_\zeta \in \mathcal{L}(T_\zeta^{1*}, c_0)$ by the rule $I_\zeta t_n^* = f_n$, where (f_n) is the canonical basis of c_0 and $(t_n^*) \subset T_\zeta^{1*}$ are the biorthogonal functionals to $(t_n) \subset T_\zeta^1$. Recall that every separable Banach space is isometrically isomorphic to a quotient of ℓ_1 (cf., e.g., [AK06, Corollary 2.3.2]). Thus, there exists a surjection $Q_\zeta : \ell_1 \rightarrow T_\zeta^{p*}$. Recall that embeddings and surjections are dual sorts of operators (cf., e.g., [Ai07, Lemma 1.30]) so that $Q_\zeta^* : T_\zeta^p \rightarrow \ell_\infty$ is an embedding. Since I_ζ^* is not class $\mathcal{JS}_{e,\zeta}$, by Proposition 4 neither is $Q_\zeta^* I_\zeta^*$. On the other hand, since T_ζ^p fails to contain a copy of ℓ_p , by Propositions 3.7 and 3.8 we have $I_\zeta^* \in \mathcal{WS}_{e,\omega_1}(\ell_p, T_\zeta^p)$. Now we can apply Theorem 3.9 to find a countable ordinal $1 \leq \alpha < \omega_1$ such that $I_\zeta^* \in \mathcal{WS}_{e,\alpha}(\ell_p, T_\zeta^p)$. It then follows from Propositions 3.3(6) and 3.8 that $Q_\zeta^* I_\zeta^* \in \mathcal{JS}_{e,\alpha}(\ell_p, \ell_\infty)$. Due to Proposition 3.3(2), this forces $\alpha > \zeta$. Letting $A_\alpha = I_\zeta Q_\zeta$ completes the proof of the claim.

Due to the claim above together with Proposition 3.3(2), for any countable ordinal $1 \leq \xi < \omega_1$ there exists a countable ordinal $\alpha > \xi$ such that $\mathcal{J}_\xi \subsetneq \mathcal{J}_\alpha$ and $\mathcal{J}_\xi^* \subsetneq \mathcal{J}_\alpha^*$. The desired chains follow from the fact that if $(\xi_i)_{i \in I}$ is any countable chain of ordinals then $\sup_i \xi_i$ is again a countable ordinal. ■

We can really just deduce the main Theorem 3.1 of this section as a corollary to the above, in light of the following elementary fact.

3.11. Proposition. *Let X and Y be Banach spaces, and let Z be a Banach space containing complemented copies of X and Y . For each closed subideal \mathcal{I} in $\mathcal{L}(X, Y)$, we define*

$$\Psi(\mathcal{I}) := [\mathcal{G}_{\mathcal{I}}](Z),$$

the closed linear span in $\mathcal{L}(Z)$ of operators factoring through elements of \mathcal{I} . Then Ψ is an order-preserving injection from the closed subideals of $\mathcal{L}(X, Y)$ into the closed ideals of $\mathcal{L}(Z)$.

Proof. This is just a minor adaptation of [Wa15, Proposition 3.9]. Let \mathcal{I} and \mathcal{J} be closed subideals in $\mathcal{L}(X, Y)$. Clearly, if $\mathcal{I} \subseteq \mathcal{J}$, then $\Psi(\mathcal{I}) \subseteq \Psi(\mathcal{J})$. Now let us suppose instead that $\Psi(\mathcal{I}) \subseteq \Psi(\mathcal{J})$, and pick any $T \in \mathcal{I}$. Let $P : Z \rightarrow \hat{X}$ and $R : Z \rightarrow \hat{Y}$ denote projections onto subspaces \hat{X} and \hat{Y} , respectively, such that there exist isomorphisms $U : X \rightarrow \hat{X}$ and $V : Y \rightarrow \hat{Y}$. Also, let $J : \hat{X} \rightarrow Z$ and $Q : \hat{Y} \rightarrow Z$ denote the corresponding embeddings, i.e. such that PJ and RQ are just identity operators acting on \hat{X} and \hat{Y} , respectively. Then $QVTU^{-1}P \in \Psi(\mathcal{I}) \subseteq \Psi(\mathcal{J})$, and so we can find a sequence of finite sums satisfying

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{m_n} B_{n,j} T_{n,j} A_{n,j} = QVTU^{-1}P,$$

where $A_{n,j} \in \mathcal{L}(Z, X)$, $B_{n,j} \in \mathcal{L}(Y, Z)$, and $T_{n,j} \in \mathcal{J}$ for all n and j . Let us set

$$S_n := \sum_{j=1}^{m_n} V^{-1} R B_{n,j} T_{n,j} A_{n,j} J U \in \mathcal{J}.$$

Then $S_n \rightarrow V^{-1} R Q V T U^{-1} P J U = T$, and since \mathcal{J} is closed we get $T \in \mathcal{J}$. This shows that $\mathcal{I} \subseteq \mathcal{J}$ as desired. ■

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